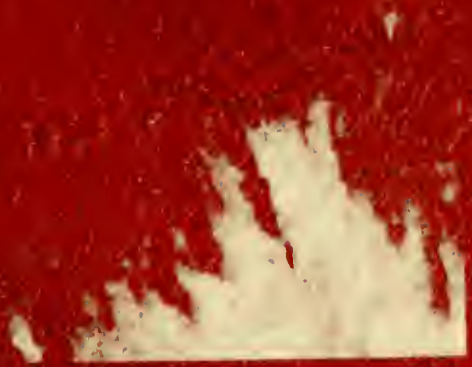

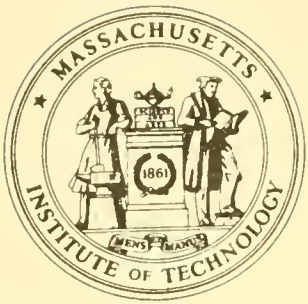

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MARSHALL AND TURVEY

ON PEAK LOAD OR JOINT PRODUCT PRICING

Paul
P. R. Kleindorfer and M. A. Crew

April 1971 530-71

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This paper is concerned with an extension of the theory of peak load or joint product pricing. It is prompted by a recent paper by Turvey and by Marshall's early analysis of joint product pricing (Marshall 1920; Turvey 1968). Both of these authors hint at solutions to a joint product or peak load problem that have not been discussed in the major contributions on peak load pricing (Steiner 1957, Williamson 1966). Both Williamson and Steiner recognize the existence of a class of peak load situations which Steiner calls the "firm peak" case. These occur where it is not possible in the off peak period to fully utilize capacity even when a price equal to marginal running cost is charged.[1] This corresponds to Marshall's "valueless straw case". When the straw is worthless, farmers concentrate on the production of a crop which has a larger proportion of ears to straw. Corresponding to this process is a firm peak situation where a public utility would install different kinds of facilities in order to vary its production methods so as to reduce the costs of servicing the peak loads. Empirical evidence of this can be noted in the electricity supply industry's production techniques, which consist of employing plants which have different cost characteristics according to their role in meeting demand. This was noted by Turvey whose paper throws some light on the problems of an electricity supply industry in meeting demand. Although Turvey does not explicitly state what are optimal peak load prices when an industry uses more than one kind of plant to meet its peak loads, he hints that prices equal to marginal running cost are somehow relevant. Turvey criticises the assumption of constant marginal running costs and constant incremental capacity costs as "too simple a notion to be meaningful". He then notes that for an electricity system consisting of plants differing in age, location, and type (and therefore also running costs), the system marginal running cost curve is upward sloping. He goes on to say: "The first consequence of this which

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is relevant here is that a price equal to marginal running costs in all off peak periods will vary between these periods." This paper will show that the relevant marginal cost for pricing decisions is not given by marginal running costs in a joint product pricing problem when it is possible to employ more than one kind of plant to meet demand.

It is a contention of this paper that a solution to the joint product or peak load problem, where there are several plants of differing costs to meet the demands, can be found, at least indirectly, in the Marshallian approach to joint product pricing. While it will be argued that a kind of marginal cost pricing is relevant, it will be shown that this is not the marginal running cost mentioned by Turvey. An idea of the kind of marginal cost involved is present in Marshall's analysis. Marshall states: "when it is possible to modify the proportions of these (joint) products, we can ascertain what part of the whole expense of production would be saved...". (Marshall 1920, p. 390). In his mathematical note XIX he explains the point further. "If in equilibrium x oxen are annually supplied and sold at a price $y = \phi(x)$; and each ox yields m units of beef; and if breeders find that by modifying the breeding and feeding of oxen they can increase their meat yielding properties to the extent of Δm units of beef (the hide and other joint products being, on balance, unaltered) and the extra expense of doing this is $\Delta y'$ then $\Delta y'/\Delta m$ represents the marginal supply price of beef; if this price were less than the selling price, it would be in the interests of the breeders to make the change" (Marshall 1920, p. 854).

The approach taken here owes something to Marshall's. In the model which will be developed below the peak period costs of using plant 2 are less than peak period price when only plant 1 is used. Because of this, in line with Marshall's solution it pays to introduce plant 2.

A Model for "Modified" Peak Load or Joint Product Pricing

It is assumed that the producer is a public utility which aims at maximizing a social welfare function, W, given by

$$W = TR + S - TC$$

where TR = total revenue

S = consumers' surplus

TC = total costs

The product is produced in two time periods, both of equal length. In each of these time periods, there is a demand curve for the output of that period. These demand curves are not identical and they are completely independent of each other.

Two plants are available for meeting the demands. Plant 1 and plant 2 have constant operating costs of b_1 and b_2 per unit per period. They have capacity costs of β_1 and β_2 per unit of capacity. It is assumed that $b_1 < b_2$, and $\beta_1 > \beta_2$. This alone is not sufficient for both plant 1 and plant 2 to be used. It is shown below that the following condition with regard to cost must be satisfied in order for both plants to be used.

(1) $$\frac{\beta_1 - \beta_2}{2} < b_2 - b_1 < \beta_1 - \beta_2$$

Let x_1 and x_2 be the quantities of output in periods 1 and 2, and let P_1 and P_2 be the corresponding market prices for these quantities. Let the quantities x_i and prices P_i for period i - 1,2, be related by the differentiable functions f_i such that $P_i = f_i(x_i)$, $f_i'(x_i) < 0$, i = 1,2, where it is assumed that $f_1(x) \geq f_2(x)$,

for all $x \geq 0$. Let $b_\ell \geq 0$ be the constant operating cost per unit of x_i per period supplied from plant $\ell = 1,2$. Let the quantity supplied from plant $\ell = 1,2$ in period $i = 1,2$ be denoted by $q_{\ell i}$ and let the capacity purchased *ab initio* on plant $\ell = 1,2$ be denoted by $\bar{q}_\ell$. Then the social welfare function, W, is given by

$$(2) \qquad W = \sum_{i=1}^{2} \{ \int_0^{x_i} f_i(y)\, dy - \sum_{\ell=1}^{2} b_\ell q_{\ell i} \} - \sum_{\ell=1}^{2} \beta_\ell \bar{q}_\ell$$

The problem to be solved is given by

$$(3) \qquad \text{Maximize } W$$
$$P_1, P_2, X_1, x_2, Q$$

subject to

$$(4) \qquad P_i = f_i(x_i) \qquad i = 1,2$$

$$(5) \qquad q_{1i} + q_{2i} = x_i \qquad i = 1,2$$

$$(6) \qquad \bar{q}_\ell - q_{\ell i} \geq 0 \qquad i = 1,2,\ \ell = 1,2$$

$$(7) \qquad P_1, P_2, x_1, x_2 \geq 0,\ Q \geq 0$$

where

$$(8) \qquad Q = (q_{11}, q_{21}, q_{12}, q_{22}, \bar{q}_1, \bar{q}_2)$$

The problem is solved in two stages. It is first noted that the minimum operating

and capacity costs for supplying specified output quantities, x_1 and x_2, in periods 1 and 2 are given by the solution to the following linear program.[2]

$$(9)\quad \text{Minimize } Y = b_1 q_{11} + b_2 q_{21} + b_1 q_{12} + b_2 q_{22} + \beta_1 \bar{q}_1 + \beta_2 \bar{q}_2$$

subject to (5), (6), and (7).

Let $Q^o(x_1, x_2)$ denote an optimal solution to (9) for specified x_1 and x_2. It can be shown that $Q^o(x_1, x_2)$ is given by the following:

$$(10)\quad Q^o = (x_2,\ x_1 - x_2,\ x_2,\ 0,\ x_2,\ x_1 - x_2) \text{ for } \frac{\beta_1 - \beta_2}{2} < b_2 - b_1 < \beta_1 - \beta_2$$

Similarly, it can be shown that if $2(b_2 - b_1) \leq \beta_1 - \beta_2$, then only plant 2 will be used; and if $\beta_1 - \beta_2 \leq b_2 - b_1$, then only plant 1 will be employed.

Returning now to the original problem (2), we may write the Lagrangian

$$(11)\quad L = W + \sum_{i=1}^{2} \{\lambda_i(q_{1i} + q_{2i} - x_i) + \eta_i(P_i - f_i(x_i)) + \sum_{i=1}^{2} \sum_{\ell=1}^{2} \mu_{\ell i}(\bar{q}_\ell - q_{\ell i})$$

The Kuhn-Tucker conditions for the problem are then given by the following:

$$(12)\quad f_i(x_i) - \eta_i f_i'(x_i) - \lambda_i \leq 0 \qquad i = 1,2$$

$$(13)\quad x_i(f_i(x_i) - \eta_i f_i'(x_i) - \lambda_i) = 0 \qquad i = 1,2$$

$$(14)\quad \eta_i \leq 0;\ \eta_i P_i = 0 \qquad i = 1,2$$

(15) $\mu_{\ell 1} + \mu_{\ell 2} - \beta_\ell \leq 0;\ \bar{q}_\ell(\mu_{\ell 1} + \mu_{\ell 2} - \beta_\ell) = 0 \qquad \ell = 1,2$

(16) $\lambda_i - \mu_{\ell i} - b_\ell \leq 0;\ q_{\ell i}(\lambda_i - \mu_{\ell i} - b_\ell) = 0 \qquad \ell = 1,2;\ i = 1,2$

(17) $\mu_{\ell i}(\bar{q}_\ell - q_{\ell i}) = 0 \qquad \ell = 1,2;\ i = 1,2$

(18) (4), (5), (6), and (7) must hold, and $\mu_{\ell i} \geq 0;\ \ell = 1,2;\ i = 1,2.$

It will be assumed in what follows that the optimal solution involves positive prices and quantities in both periods, so that from (12) and (14), $\eta_i = 0$, $f_i(x_i) = \lambda_i$. Since marginal costs are non-decreasing, it is clear from the assumed form of the demand curves that $x_1 \geq x_2$ at the optimum.

Case i (Firm Peak Case): $x_1 > x_2 > 0$. In this case (10) yields Q as

(19) $Q = (x_2,\ x_1 - x_2,\ x_2,\ 0,\ x_2,\ x_1 - x_2).$

Now since $q_{22} = 0 < x_1 - x_2 = \bar{q}_2$, we have by (17) that $\mu_{22} = 0$. But $\bar{q}_2 > 0$ yields from (15) that $\mu_{21} + \mu_{22} = \beta_2$. Furthermore, $q_{21} = x_1 - x_2 > 0$ implies by (16) that $\lambda_1 = \mu_{21} + b_2$ so that $\lambda_1 = b_2 + \beta_2$. Since (12) and (14) imply, with the assumption the $P_i > 0$, $P_i = f_i(x_i) = \lambda_i$, the optimal price, P_1, is given by

(20) $P_1 = b_2 + \beta_2.$

To obtain P_2 we note that since $q_{11} = q_{12} = x_2 > 0$, it follows by (16) that $\lambda_i = \mu_{1i} + b_1$ for i = 1,2. Therefore,

(21) $\lambda_1 + \lambda_2 = \mu_{11} + \mu_{12} + 2b_1$

Now $\bar{q}_1 = x_2 > 0$ implies by (15) that $\mu_{11} + \mu_{12} = \beta_1$, so that (21) yields $\lambda_1 + \lambda_2 = 2b_1 + \beta_1$. Using $P_i = \lambda_i$, $i = 1,2$, and (20) it follows that

(22) $P_2 = 2b_1 + \beta_1 - (b_2 + \beta_2)$

It may easily be verified from (1) that $b_1 < P_2 < b_2$ and $\beta_2 < P_1 < \beta_1$.

Thus, if $x_1 > x_2 > 0$, $i = 1,2$, then the P_i are given by (20) and (22) with x_i determined by $P_i = f_i(x_i)$ and Q given by (19).

Case ii (Shifting Peak Case)[3]: $x_1 = x_2 = x > 0$. In this case we have from (10) that

(23) $Q = (x, 0, x, 0, x, 0)$

Moreover, (12) and (14) yield as above:

(24) $P_i = \lambda_i = f_i(x)$

Now $q_{11} = q_{12} = \bar{q}_1 > 0$ implies by (15) and (16) that $\mu_{11} + \mu_{12} = \beta_1$ and $\lambda_i = \mu_{1i} + b_1$, $i = 1,2$; so that $\lambda_1 + \lambda_2 = \mu_{11} + \mu_{12} + 2b_1$; or, using (24),

(25) $f_1(x) + f_2(x) = 2b_1 + \beta_1$

Thus, if $x_1 = x_2 = x > 0$, then x is determined by (25) with Q and P_i given by (23) and (24).

It is now necessary to determine when case i or case ii applies. It should be noted by (15) and (16) that the optimum must satisfy

$$(26) \qquad \lambda_1 \leq \mu_{21} + b_2 \leq b_2 + \beta_2 - \mu_{22},$$

and since, by (18) $\mu_{22} \geq 0$, it follows that

$$(27) \qquad \lambda_1 \leq b_2 + \beta_2.$$

But in both case i and case ii, $\lambda_i = f_i(x_i)$ and $f_1(x_1) + f_2(x_2) = 2b_1 + \beta_1$. Therefore, since $f_i(x_i) = P_i \geq 0$, it follows from (20), (22), (25), and (27) that

$$(28) \qquad P_1 \leq b_2 + \beta_2,\ P_2 \geq 2b_1 + \beta_1 - (b_2 + \beta_2)$$

with equality in (28) if $x_1 > x_2 > 0$. It can now be shown that cases i and ii are mutually exclusive. Assume that there exist x_1, x_2, and x such that $x_1 \leq x_2$ and

$$(29) \qquad f_1(x) \leq b_2 + \beta_2 = f_1(x_1)$$

$$(30) \qquad f_2(x) \geq 2b_1 + \beta_1 - (b_2 + \ _2) = f_2(x_2)$$

Then since $f_i' < 0$, it follows from (29) and (30) that $x_1 \leq x_2$, a contradiction. Thus, the solution outlined above is unique, at least under the assumption that prices and quantities demanded in both periods are positive.

Implications of the Model

By proving that the two-plant solution will only apply to the firm peak case

Marshall's point on valueless straw is emphasized. The above analysis indicates that a welfare maximizing public utility facing a firm peak situation should move in the long-run towards the installation of plants of various types when appropriate marginal running and capacity costs relationships are fulfilled. The prices in this case differ from those applicable in the one plant case.

The solution has certain properties which are worth stressing. Price is not set equal to marginal running costs when two plants are used. This is quite different from the one plant firm peak case where in the off peak period price is set equal to marginal cost. Price is set equal to the marginal cost of expansion of the off peak quantity (on the assumption that peak demand is also expanded in the same proportion). The relevant dual variable is λ_i, (which is equal to price), and the value of this changes according to the cost and demand conditions.

The point can be illustrated by contrasting the one plant case and two plant case. In the simple case where there is initially only plant 1 available $P_1 = \lambda_1 = b_1 + \lambda_1$ and $P_2 = \beta_2 = b_1$.[4] λ_2 is the marginal cost of expansion in the off peak period. It is clearly marginal running cost as $\lambda_2 = b_1 = 2b_1 + \beta_1 - \lambda_1$. Verbally, with constant costs, marginal cost of the off peak commodity is equal to the per unit costs of supplying output in both periods as given by the demand curves less the value of the commodity sold in the peak period.[5]

In the modified case described by the model presented above exactly the same rule is applicable. Price in both periods is set equal to marginal cost. In the off peak period marginal cost is given by the marginal cost of supplying output in both periods, $2b_1 + \beta_1$, less the value of the peak period output, $b_2 + \beta_2$. This is clearly not equal to marginal running costs (either b_1 or b_2). It corresponds much more closely to Marshall's ideas on marginal cost in a joint product pricing problem, as marginal cost is only defined by reference to the demand curves for each and every joint product.

It is useful to state a few implications of the choice of the social welfare function. Turvey did not assume an explicit social welfare function. As most of his discussion and illustrations are concerned with the electricity supply industry in England and Wales, it is probable that he had before him the targets which the British Government sets for Nationalized Industries. These are not explicitly the maximization of the excess of consumer surplus and total revenue over cost. They consist of meeting a certain rate of return.[6] It may be argued - perhaps not too strongly - that the objectives of Nationalized Industries are more closely related to the Social Welfare Function of this paper than they are to profit maximization. In any case the difference in the objectives of the Nationalized Industry is not in itself sufficient to explain the difference between the pricing solution outlined here and the pricing solution implied by Turvey.

This point can be illustrated further by brief examination of the effects of adopting the profit maximization assumption.[7] The problem may be stated as follows:

$$\text{Maximize } \Pi = \sum_{i=1}^{2} \{x_i f_i(x_i) - \sum_{\ell=1}^{2} b_\ell q_\ell\} - \sum_{\ell=1}^{2} \beta_\ell \bar{q}_\ell$$

subject to:

$$P_i = f_i(x_i) \qquad i = 1,2;$$

$$q_{1i} + q_{2i} = x_i \qquad i = 1,2;$$

$$\bar{q}_\ell - q_{\ell i} \geq 0 \qquad i = 1,2,\ \ell = 1,2;$$

$$P_1, P_2, x_1, x_2 \geq 0,\ Q \geq 0.$$

It is first noted that the same results, (10), hold for $Q^{\circ}(x_1, x_2)$. Proceeding as above, define the Lagrangian

$$L = \Pi + \sum_{i=1}^{2} \lambda_i (q_{1i} + q_{2i} - x_i) + \eta_i(P_i - f_i(x_i)) + \sum_{i=1}^{2} \sum_{\ell=1}^{2} \mu_{\ell i}(\bar{q}_\ell - q_{\ell i}).$$

Conditions (14) through (18) are the same in this case as may be verified by considering the appropriate Kuhn-Tucker conditions. Instead of (12) and (13), the following conditions for x_i are obtained:

$$(12^*) \qquad f_i(x_i) + f_i'(x_i)x_i - \eta_i f_i'(x_i) - \lambda_i \leq 0$$

$$(13^*) \qquad x_i(f_i(x_i) + f_i'(x_i)x_i - \eta_i f_i'(x_i) - \lambda_i) = 0$$

In particular, assuming positive quantities and prices, it follows from (12^*), (13^*) and (14) that

$$\lambda_i = \frac{d}{dx_i}(x_i f_i(x_i)) \text{ or } \lambda_i = MR_i.$$

Thus, the fundamental results differ only to the extent that the dual variables have to be regarded as marginal revenues and price set accordingly.

The normative rule presented here will also describe the equilibrium results of a purely competitive decentralized process provided the conditions of production are the same for the public utility monopoly as they are for the competitive industry.[9]

Conclusions

The major point of this analysis was to develop a solution to the peak load

problem where two plants were employed, unlike the one plant case which had been the primary theoretical model of interest until Turvey's recent paper. It was demonstrated how the solution here differed from the one implied by Turvey, where prices were apparently to be set equal to marginal running cost. No simple cost-based solution proved possible. The solution developed here, in this sense, is similar to the early contributions to joint product pricing of Marshall in that demands have to be brought explicitly into the analysis. Once this is done a pricing rule based upon the dual variables is possible. This rule applies with the usual modifications not only to the welfare maximizing monopoly described here and to pure competition, but also to pure (profit maximizing) monopoly.

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FOOTNOTES

[1]Marginal running cost may possibly be zero. In the Marshallian case of joint products it is zero. Indeed, what differentiates later peak load problems from the Marshallian analysis is the introduction of the marginal running costs.

[2]The solution to this problem (see equation (10) below) is derived in a separate appendix available from the authors.

[3]This terminology was used by Steiner (1957). It simply refers to the case where it is possible to utilize plant fully in both periods. Steiner coined the phrase from the fact that if the firm peak prices are employed in this case the peak period quantity demanded becomes less than the off peak period quantity demanded. Thus, he argued, the peak "shifts".

[4]As $\bar{q}_2$, q_{21}, q_{22}, are all zero, it follows from (17) that $\mu_{22} = 0$ and $\mu_{21} = 0$. Therefore from (16) $\lambda_1 = b_1 + \mu_{11}$ with $q_{11} > 0$ and $\lambda_2 = b_1 + \mu_{12}$ with $q_{12} > 0$. From (15) $\mu_{11} + \mu_{12} = \beta_1$; therefore $\lambda_1 = \beta_1 + b_1 - \mu_{12}$. Since in the firm peak case $q_1 - q_{12} > 0$ and from (17), $\mu_{12} = 0$, it follows that $\lambda_1 = \beta_1 + b_1$ and from (16) that $\lambda_2 = b_1$.

[5]Weil (1968) has an interesting analysis of this problem which makes basically the same point, while examining the beef and hides joint product pricing problem. In his example there is no complication of running costs. He simply has 10 units to correspond to the $2b_1 + \beta_1$ of this analysis. However, his result is just the same. His price for beef is the cost of the cow as given by the demand curves, less the

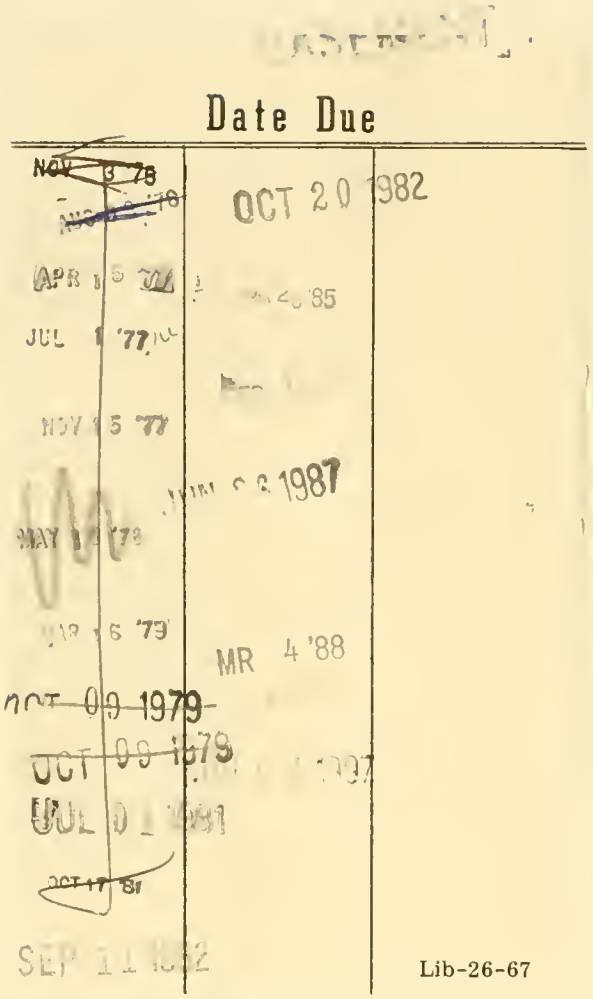
value of the hides - in terms of his example $\lambda_1 = 10 - \lambda_2$. S. C. Littlechild has made a similar point on the relevance of the Marshallian analysis to the problem of production over time, Littlechild (1970).

[6]The criteria for performance are to be found in two British Government Publications (1961, 1967).

[7]Profit maximization was at the heart of Marshall's approach. For example, it is implied by the last sentence of the mathematical note XIX.

[8]This point is made clearly by Weil (1968), who describes his multipliers as marginal revenues. In footnote 5 this point was ignored, because it makes no difference to the fundamental principles involved. Irrespective of whether profits or W are maximized in Weil's problem the two dual variables sum to the same amount. The only difference is that in Weil's case they are marginal revenues while in the case described in footnote 5 they are prices (which are equal to marginal costs as illustrated in the analysis). Profit maximizing simply requires that demands be incorporated explicitly into the analysis through the marginal revenue functions instead of the demand functions. This applies to both the firm and shifting peak results. For the firm peak case, where $x_1 > x_2 > 0$ and $Q = (x_2, x_1 - x_2, x_2\ 0, x_2, x_1 - x_2)$, the results are that $\lambda_1 = b_2 + \beta_2$ and $\lambda_2 = 2b_1 + \beta_1 - (b_2 + \beta_2)$. For the shifting peak case, with $x_1 = x_2 = x > 0$ and $Q = (x, 0, x, 0, x, 0)$, $\lambda_1 + \lambda_2 = 2b_1 + \beta_1$.

[9]See Officer (1966). Actually constant returns to scale prevail and so the result applies with the usual qualification about constant returns and the "indeterminacy of the purest competition", Samuelson (1947).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